# Even-homogeneous supermanifolds on the complex projective line [1]

**E.G. Vishnyakova**

Abstract. We obtain the classification of even-homogeneous non-split complex supermanifolds of dimension $1|m$, $m \leq 3$, on $\mathbb{CP}^1$, up to isomorphism. For $m = 2$, we show that there exists only one such supermanifold, which is the superquadric in $\mathbb{CP}^{2|2}$ constructed independently by P. Green [4] and V.P. Palamodov [1]. For $m = 3$, we prove that there exists a series of non-split even-homogeneous supermanifolds, parameterized by elements in $\mathbb{Z} \times \mathbb{Z}$, three series of non-split even-homogeneous supermanifolds, parameterized by elements in $\mathbb{Z}$, and finite set of exceptional supermanifolds.

**1. Introduction.** The study of homogeneous supermanifolds, i.e. supermanifolds which possess transitive actions of Lie supergroups, with underlying manifold $\mathbb{CP}^1$ was started in [2]. There the classification of homogeneous complex supermanifolds of dimension $1|m$, $m \leq 3$, up to isomorphism was given. It was proven that in the case $m = 2$ there exists only one non-split homogeneous supermanifold constructed by P. Green [4] and V.P. Palamodov [1]. For $m = 3$ it was shown that there exists a series of non-split homogeneous supermanifolds, parameterized by $k = 0, 2, 3, \cdots$.

The purpose of our paper is to classify up to isomorphism even-homogeneous non-split complex supermanifolds of dimension $1|m$, $m \leq 3$, on $\mathbb{CP}^1$. (A supermanifold is called homogeneous if it possesses a transitive action of a Lie group. It is easy to see that any homogeneous supermanifold is even-homogeneous. The converse statement does not hold true.) For $m \geq 4$, the classification of (even-) homogeneous supermanifolds is not completed yet. The first reason for this is that in the case $m \geq 4$ the Green moduli space of non-isomorphic supermanifolds is given by the non-abelian cohomology set modulo a certain group action, which is difficult to compute explicitly. The second reason is that we as in [2] use Theorem 2 to prove the existence of a transitive action of a Lie (super)group on a supermanifold. The statement of this theorem does not hold true in the case $m \geq 4$, see [2] for more details. We notice that a 1-parameter family of mutually non-isomorphic non-split

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homogeneous supermanifolds of dimension $1|4$ with the reduction $\mathbb{CP}^1$ was obtained in [3]. Some other classification results concerning non-split complex homogeneous supermanifolds on $\mathbb{CP}^n$ of dimension $n|m$, $m \leq n$, can be found in [7, 8].

The paper is structured as follows. In Section 2 we explain the idea of the classification. A similar idea was used in [2] by the classification of homogeneous supermanifolds on $\mathbb{CP}^1$. In Section 3 we calculate the 1-cohomology group with values in the tangent sheaf. We use here an easier way than in [2], which allows to classify supermanifolds under less restrictive assumptions than in [2].

By the Green Theorem we can assign a supermanifold to each cohomology class of the 1-cohomology group. In Section 4 we find out cohomology classes corresponding to even-homogeneous supermanifolds. Notice that these supermanifolds can be isomorphic. The classification up to isomorphism of even-homogeneous complex supermanifolds of dimension $1|m$, $m \leq 3$, on $\mathbb{CP}^1$ is obtained in Section 5.

Results of the paper may have applications to the string theory, see [9] for more details.

**2. Even-homogeneous supermanifolds on $\mathbb{CP}^1$.** We study complex analytic supermanifolds in the sense of [2, 5]. If $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ is a supermanifold, we denote by $\mathcal{M}_0$ the *underlying complex manifold* of $\mathcal{M}$ and by $\mathcal{O}_\mathcal{M}$ the *structure sheaf* of $\mathcal{M}$, i.e. the sheaf of commutative associative complex superalgebras on $\mathcal{M}_0$. Denote by $\mathcal{T}_\mathcal{M}$ the *tangent sheaf* of $\mathcal{M}$, i.e. the sheaf of derivations of the structure sheaf $\mathcal{O}_\mathcal{M}$. Denote by $(\mathcal{T}_\mathcal{M})_{\bar{0}} \subset \mathcal{T}_\mathcal{M}$ the subsheaf of all even vector fields. An *action of a Lie group $G$ on a supermanifold* $\mathcal{M}$ is a morphism $\nu = (\nu_0, \nu^*) : G \times \mathcal{M} \to \mathcal{M}$ such that it satisfies the usual conditions, modeling the action axioms. An action $\nu$ is called *even-transitive* if $\nu_0$ is transitive. A supermanifold $\mathcal{M}$ is called *even-homogeneous* if it possesses an even-transitive action of a Lie group.

Assume that $\mathcal{M}_0$ is compact and connected. It is well-known that the group of all automorphisms of $\mathcal{M}$, which we denote by $\operatorname{Aut}\mathcal{M}$, is a Lie group with the Lie algebra $H^0(\mathcal{M}_0, (\mathcal{T}_\mathcal{M})_{\bar{0}})$. (Recall that by definition any morphism of a supermanifold is even.) Let us take any homomorphism of Lie algebras $\varphi : \mathfrak{g} \to H^0(\mathcal{M}_0, (\mathcal{T}_\mathcal{M})_{\bar{0}})$. We can assign the homomorphism of Lie groups $\Phi : G \to \operatorname{Aut}\mathcal{M}$ to $\varphi$, where $G$ is the simple connected Lie group with the Lie algebra $\mathfrak{g}$. Notice that $\Phi$ is even-transitive iff the image of $\mathfrak{g}$ in $H^0(\mathcal{M}_0, (\mathcal{T}_\mathcal{M})_{\bar{0}})$ generates the tangent space $T_x(\mathcal{M})$ at any point $x \in \mathcal{M}_0$.

In this paper we will consider the case $\mathcal{M}_0 = \mathbb{CP}^1$. Therefore, the classification problem reduces to the following problem: *to classify up to isomorphism complex supermanifolds $\mathcal{M}$ of dimension $1|m$, $m \leq 3$, such that*

$H^0(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_{\bar{0}})$ *generates the tangent space* $T_x(\mathcal{M})$ *at any point* $x \in \mathcal{M}_0$.

Recall that a supermanifold $\mathcal{M}$ is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$, where $\mathcal{E}$ is a sheaf of sections of a vector bundle $\mathbf{E}$ over $\mathcal{M}_0$. In this case $\dim \mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and $m$ is the rank of $\mathbf{E}$. The structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a split supermanifold possesses by definition the $\mathbb{Z}$-grading; it induces the $\mathbb{Z}$-grading in $\mathcal{T}_{\mathcal{M}} = \bigoplus_{p=-1}^{m} (\mathcal{T}_{\mathcal{M}})_p$. Hence, the superspace $H^0(\mathcal{M}_0, \mathcal{T}_{\mathcal{M}})$ is also $\mathbb{Z}$-graded. Consider the subspace $\operatorname{End} \mathbf{E} \subset H^0(\mathcal{M}_0, \mathcal{T}_{\mathcal{M}})_0$ consisting of all endomorphisms of the vector bundle $\mathbf{E}$, which induce the identity morphism on $\mathcal{M}_0$. Denote by $\operatorname{Aut} \mathbf{E} \subset \operatorname{End} \mathbf{E}$ the group of automorphisms containing in $\operatorname{End} \mathbf{E}$. We define an action Int of $\operatorname{Aut} \mathbf{E}$ on $\mathcal{T}_{\mathcal{M}}$ by $\operatorname{Int} A : v \mapsto A v A^{-1}$. Since the action preserves the $\mathbb{Z}$-grading, we have the action of $\operatorname{Aut} \mathbf{E}$ on $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathcal{M}})_2)$.

We can assign the split supermanifold $\operatorname{gr} \mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\operatorname{gr} \mathcal{M}})$ to each supermanifold $\mathcal{M}$, see e.g. [2]. It is called the *retract* of $\mathcal{M}$. To classify supermanifolds, we will use the following corollary of the well-known Green Theorem (see e.g. [2] for more details).

**Theorem 1.** [Green] *Let* $\widetilde{\mathcal{M}} = (\mathcal{M}_0, \bigwedge \mathcal{E})$ *be a split supermanfold of dimension* $n|m$*, where* $m \leq 3$*. Then classes of isomorphic supermanifolds* $\mathcal{M}$ *with the retract* $\operatorname{gr} \mathcal{M} = \widetilde{\mathcal{M}}$ *are in bijection with orbits of the action* Int *of the group* $\operatorname{Aut} \mathbf{E}$ *on* $H^1(\mathcal{M}_0, (\mathcal{T}_{\widetilde{\mathcal{M}}})_2)$.

**Remark.** This theorem allows to classify supermanifolds $\mathcal{M}$ such that $\operatorname{gr} \mathcal{M}$ is fixed up to isomorphisms which induce identity morphism on $\operatorname{gr} \mathcal{M}$.

In what follows we will consider the case $\mathcal{M}_0 = \mathbb{CP}^1$. Let $\mathcal{M}$ be a supermanifold of dimension $1|m$. Denote by $U_0$ and $U_1$ the standard charts on $\mathbb{CP}^1$ with coordinates $x$ and $y = \frac{1}{x}$ respectively. By the Grothendieck Theorem we can cover $\operatorname{gr} \mathcal{M}$ by two charts $(U_0, \mathcal{O}_{\operatorname{gr} \mathcal{M}}|_{U_0})$ and $(U_1, \mathcal{O}_{\operatorname{gr} \mathcal{M}}|_{U_1})$ with local coordinates $x$, $\xi_1, \ldots, \xi_m$ and $y$, $\eta_1, \ldots, \eta_m$, respectively, such that in $U_0 \cap U_1$ we have

$$y = x^{-1}, \quad \eta_i = x^{-k_i} \xi_i, \ i = 1, \ldots, m,$$

where $k_i, \ i = 1, \ldots, m$, are integers. We will identify $\operatorname{gr} \mathcal{M}$ with the set $(k_1, \ldots, k_m)$. Note that a permutation of $k_i$ induces the automorphism of $\operatorname{gr} \mathcal{M}$. It was shown that any supermanifold $\operatorname{gr} \mathcal{M}$ is even-homogeneous, see [2], Formula (18). The following theorem was also proven in [2], Proposition 14:

**Theorem 2.** *Assume that* $m \leq 3$ *and* $\mathcal{M}_0 = \mathbb{CP}^1$*. Let* $\mathcal{M}$ *be a supermanifold with the retract* $\operatorname{gr} \mathcal{M} = \bigwedge \mathcal{E}$*, which corresponds to the cohomology class* $\gamma \in H^1(\mathcal{M}_0, (\mathcal{T}_{\operatorname{gr} \mathcal{M}})_2)$ *by Theorem 1. The following conditions are equivalent:*

1. *The supermanifold* $\mathcal{M}$ *is even-homogeneous.*

2. *There is a subalgebra* $\mathfrak{a} \simeq \mathfrak{sl}_2(\mathbb{C})$ *such that*

$$H^0(\mathcal{M}_0, (\mathcal{T}_{\mathrm{gr}\,\mathcal{M}})_0) = \operatorname{End}\mathbf{E} \niplus \mathfrak{a}, \tag{1}$$

*and* $[v, \gamma] = 0$ *in* $H^1(\mathcal{M}_0, (\mathcal{T}_{\mathrm{gr}\,\mathcal{M}})_2)$ *for all* $v \in \mathfrak{a}$.

Here $\mathbf{E}$ is the vector bundle corresponding to the locally free sheaf $\mathcal{E}$.

From now on we will omit the index $\mathrm{gr}\,\mathcal{M}$ and will denote by $\mathcal{T}$ the sheaf of derivations of $\mathcal{O}_{\mathrm{gr}\,\mathcal{M}}$. Recall that the sheaf $\mathcal{O}_{\mathrm{gr}\,\mathcal{M}}$ is $\mathbb{Z}$-graded; it induces the $\mathbb{Z}$-grading in $\mathcal{T} = \bigoplus_p \mathcal{T}_p$. Denote by $H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{a}} \subset H^1(\mathbb{CP}^1, \mathcal{T}_2)$ the subset of $\mathfrak{a}$-invariants, i.e. the set of all elements $w$ such that $[v, w] = 0$ for all $v \in \mathfrak{a}$. The supermanifold corresponding to a cohomology class $\gamma \in H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{a}}$ by Theorem 1 is called $\mathfrak{a}$-*even-homogeneous.*

The description of subalgebras $\mathfrak{a}$ satisfying (1) up to conjugation by elements from $\operatorname{Aut}\mathbf{E}$ and up to renumbering of $k_i$ was obtained in [2]:
1) $\mathfrak{a} = \mathfrak{s} = \langle \mathbf{e} = \frac{\partial}{\partial x}, \mathbf{f} = \frac{\partial}{\partial y}, \mathbf{h} = [\mathbf{e}, \mathbf{f}]\rangle$.
2) $\mathfrak{a} = \mathfrak{s}' = \langle \mathbf{e}' = \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial \xi_1}, \mathbf{f}' = \frac{\partial}{\partial y} + \eta_1 \frac{\partial}{\partial \eta_2}, \mathbf{h}' = [\mathbf{e}', \mathbf{f}']\rangle$ if $k_1 = k_2$.
3) $\mathfrak{a} = \mathfrak{s}'' = \langle \mathbf{e}'' = \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial \xi_1} + \xi_3 \frac{\partial}{\partial \xi_2}, \mathbf{f}'' = \frac{\partial}{\partial y} + 2\eta_1 \frac{\partial}{\partial \eta_2} + 2\eta_2 \frac{\partial}{\partial \eta_3}, \mathbf{h}'' = [\mathbf{e}'', \mathbf{f}'']\rangle$ if $k_1 = k_2 = k_3$.

**3. Basis of** $H^1(\mathbb{CP}^1, \mathcal{T}_2)$. Assume that $m = 3$. Let $\mathcal{M}$ be a split supermanifold, $\mathcal{M}_0 = \mathbb{CP}^1$ be its reduction and $\mathcal{T}$ be its tangent sheaf. In [2] the $\mathfrak{s}$-invariant decomposition

$$\mathcal{T}_2 = \sum_{i<j} \mathcal{T}_2^{ij} \tag{2}$$

was obtained. The sheaf $\mathcal{T}_2^{ij}$ is a locally free sheaf of rank 2; its basis sections over $(U_0, \mathcal{O}_{\mathcal{M}}|_{U_0})$ are:

$$\xi_i\xi_j \frac{\partial}{\partial x}, \quad \xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l}; \tag{3}$$

where $l \neq i, j$. In $U_0 \cap U_1$ we have

$$\begin{aligned} \xi_i\xi_j \frac{\partial}{\partial x} &= -y^{2-k_i-k_j}\eta_i\eta_j \frac{\partial}{\partial y} - k_l y^{1-k_i-k_j}\eta_i\eta_j\eta_l \frac{\partial}{\partial \eta_l}, \\ \xi_i\xi_j\xi_l \frac{\partial}{\partial \xi_l} &= y^{-k_i-k_j}\eta_i\eta_j\eta_l \frac{\partial}{\partial \eta_l}. \end{aligned} \tag{4}$$

Let us calculate a basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$. We will use the Čech cochain complex of the cover $\mathfrak{U} = \{U_0, U_1\}$. Hence, 1-cocycle with values in the sheaf $\mathcal{T}_2^{ij}$ is a section $v$ of $\mathcal{T}_2^{ij}$ over $U_0 \cap U_1$. We are looking for *basis cocycles*, i.e. cocycles such that their cohomology classes form a basis of $H^1(\mathfrak{U}, \mathcal{T}_2^{ij}) \simeq H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$. Note that if $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is holomorphic in $U_0$ or $U_1$ then the cohomology class of $v$ is equal to 0. Obviously, any $v \in Z^1(\mathfrak{U}, \mathcal{T}_2^{ij})$ is a linear combination of vector fields (3) with holomorphic in $U_0 \cap U_1$ coefficients. Further, we expand these coefficients in a Laurent series in $x$ and drop the summands $x^n$, $n \geq 0$, because they are holomorphic in $U_0$. We see that $v$ can be replaced by

$$v = \sum_{n=1}^{\infty} a_{ij}^n x^{-n} \xi_i \xi_j \frac{\partial}{\partial x} + \sum_{n=1}^{\infty} b_{ij}^n x^{-n} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}, \tag{5}$$

where $a_{ij}^n, b_{ij}^n \in \mathbb{C}$. Using (4), we see that the summands corresponding to $n \geq k_i + k_j - 1$ in the first sum of (5) and the summands corresponding to $n \geq k_i + k_j$ in the second sum of (5) are holomorphic in $U_1$. Further, it follows from (4) that

$$x^{2-k_i-k_j} \xi_i \xi_j \frac{\partial}{\partial x} \sim -k_l x^{1-k_i-k_j} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}.$$

Hence the cohomology classes of the following cocycles

$$\begin{aligned} &x^{-n} \xi_i \xi_j \frac{\partial}{\partial x}, \quad n = 1, \ldots, k_i + k_j - 3, \\ &x^{-n} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}, \quad n = 1, \ldots, k_i + k_j - 1, \end{aligned} \tag{6}$$

generate $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$. If we examine linear combination of (6) which are cohomological trivial, we get the following theorem.

**Theorem 3.** *Assume that $i < j$, $l \neq i, j$. The basis of $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij})$*

1. *is given by* (6) *if $k_i + k_j > 3$;*
2. *is given by*

$$x^{-1} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l}, \quad x^{-2} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l},$$

*if $k_i + k_j = 3$;*

3. *is given by*

$$x^{-1} \xi_i \xi_j \xi_l \frac{\partial}{\partial \xi_l},$$

*if $k_i + k_j = 2$, $k_l = 0$.*

4. *If $k_i + k_j = 2$, $k_l \neq 0$ or $k_i + k_j < 2$, we have $H^1(\mathbb{CP}^1, \mathcal{T}_2^{ij}) = \{0\}$.*

Note that the similar method can be used for computation of a basis of $H^1(\mathbb{CP}^1, \mathcal{T}_q)$ for any $m$ and $q$.

**4. Basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{a}}$.** Let us calculate a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$. The decomposition (2) is $\mathfrak{s}$-invariant, hence,

$$H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}} = \bigoplus_{i<j} H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})^{\mathfrak{s}}.$$

Denote by $[z]$ the cohomology class corresponding to a 1-cocycle $z$.

**Theorem 4.** *Let us fix $i<j$ and $l\neq i,j$. Then*
1) $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})^{\mathfrak{s}} = \langle[\frac{1}{x}\xi_i\xi_j\frac{\partial}{\partial x} + \frac{k_l}{2x^2}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}]\rangle$ *if* $k_i+k_j=4$,
2) $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})^{\mathfrak{s}} = \langle[\frac{1}{x}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}]\rangle$ *if* $k_i+k_j=2$, $k_l=0$,
3) $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})^{\mathfrak{s}} = \{0\}$ *otherwise.*

*Proof.* We have to find out highest vectors of the $\mathfrak{s}$-module $H^1(\mathbb{CP}^1,\mathcal{T}_2^{ij})$ having weight 0. By Propositions 8 and 9 of [2], any cocycle $z$ from the Theorem 3 fulfils the condition $[\mathbf{h},z]=\lambda z$. More precisely, $\lambda=0$ if $z = x^{-r}\xi_i\xi_j\frac{\partial}{\partial x}$, $2r=k_i+k_j-2$ $z=x^{-r}\xi_i\xi_j\xi_l\frac{\partial}{\partial \xi_l}$, $2r=k_i+k_j$. If we examine a linear combination $w$ of these cocycles such that $[\mathbf{e},w]\sim 0$, we obtain the result of the Theorem.□

**Theorem 5.** *Assume that $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}\neq 0$. Then we have the following possibilities:*
1) $(k_1,k_2,k_3)=(2,2,1)$ *and a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}$ is given by*

$$[\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{1}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}],\ \ [\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2} - \frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1}]; \tag{7}$$

2) $(k_1,k_2,k_3)=(2,2,3)$ *and a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}$ is given by*

$$\begin{gathered}[\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x} + \tfrac{3}{2x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3}],\\ [\tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x} + \tfrac{1}{x^2}\xi_1\xi_3\tfrac{\partial}{\partial x} + \tfrac{2}{3x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1} - \tfrac{4}{3x^3}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2}];\end{gathered} \tag{8}$$

3) $(k_1,k_2,k_3)=(2,2,k_3)$, $k_3\neq 1,3$; $(k_1,k_2,k_3)=(k,k,3-k)$, $k\neq 2$ *or* $(k_1,k_2,k_3)=(k,k,5-k)$, $k\neq 2$ *or* $(k_1,k_2,k_3)=(1,1,0)$. *Then*

$$\dim H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'} = 1$$

*and a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}$ is given by the following cocycles:*

$$\begin{gathered}[\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x} + \tfrac{k_3}{2x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3}],\ [\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2} - \tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1}],\\ [\tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x} + \tfrac{1}{x^2}\xi_1\xi_3\tfrac{\partial}{\partial x} + \tfrac{k}{3x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1} - \tfrac{2k}{3x^3}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2}],\\ [\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3}],\end{gathered} \tag{9}$$

*respectively.*

*Proof.* Use a similar argument as in Theorem 4.□

The calculation of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$ and $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}$ was already done in [2], Proposition 19 and Proposition 21, using more difficult methods. Note that the case 2 of Theorem 4 and the case $(k_1,k_2,k_3)=(1,1,0)$ of Theorem 5 was lost in [2]. Furthermore, in [2] the following theorem was proven, see Proposition 22.

**Theorem 6.** *Assume that $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}''}\neq 0$. Then we have the following possibilities:*
1) $(k_1,k_2,k_3)=(2,2,2)$ *and the basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}''}$ is given by*

$$[\frac{1}{x^3}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}-\frac{1}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}+\frac{1}{2x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}]; \tag{10}$$

2) $(k_1,k_2,k_3)=(3,3,3)$ *and the basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}''}$ is given by*

$$\begin{array}{c}[\frac{1}{x^3}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{1}{2x^2}\xi_1\xi_3\frac{\partial}{\partial x}+\frac{1}{2x}\xi_2\xi_3\frac{\partial}{\partial x}\\ +\frac{3}{8x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}-\frac{3}{4x^3}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}+\frac{9}{4x^4}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}].\end{array} \tag{11}$$

## 5. Classification of even-homogeneous supermanifolds

In Section 4 we calculated a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$ and $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}'}$ and gave a basis of $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}''}$, which were calculated in [2]. In this section we will complete the classification of even-homogeneous supermanifolds, i.e. we will find out, which vectors of these spaces belong to different orbits of the action of $\operatorname{Aut}\mathbf{E}$ on $H^1(\mathbb{CP}^1,\mathcal{T}_2)$.

Let $(\xi_i)$ be a local basis of $\mathbf{E}$ over $U_0$ and $A$ be an automorphism of $\mathbf{E}$. Assume that $A(\xi_j)=\sum a_{ij}(x)\xi_i$. In $U_1$ we have

$$A(\eta_j)=A(y^{k_j}\xi_j)=\sum y^{k_j-k_i}a_{ij}(y^{-1})\eta_i.$$

Therefore, $a_{ij}(x)$ is a polynomial in $x$ of degree no greater than $k_j-k_i$, if $k_j-k_i\geq 0$ and 0, if $k_j-k_i<0$. We will denote by $b_{ij}$ the entries of the matrix $B=A^{-1}$. The entries are also polynomials in $x$ of degree no greater than $k_j-k_i$. We will need the following formulas:

$$\begin{array}{c}A(\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_k})A^{-1}=\det(A)\sum_s b_{ks}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_s};\\ A(\xi_i\xi_j\frac{\partial}{\partial x})A^{-1}=\det(A)\sum_{k<s}(-1)^{l+r}b_{lr}\xi_k\xi_s\frac{\partial}{\partial x}+\\ +\det(A)\sum_s b'_{ls}\xi_i\xi_j\xi_l\frac{\partial}{\partial\xi_s}.\end{array} \tag{12}$$

where $i<j$, $l\neq i,j$, $r\neq k,s$ and $b'_{ls}=\frac{\partial}{\partial x}(b_{ls})$.

**Theorem 7.** [Classification of $\mathfrak{s}$-even-homogeneous supermanifolds.]

1. *Assume that*

$$\{k_1, 4-k_1, k_3\} \neq \{-2,0,4\}, \quad \{k, 2-k, 0\} \neq \{-2,0,4\}$$

*as sets. Then there exists a unique up to isomorphism $\mathfrak{s}$-even-homogeneous non-split supermanifold with retract*

***a.*** $(k_1, 4-k_1, k_3)$*, which correspond to the cocycle*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{k_3}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3};$$

***b.*** $(k, 2-k, 0)$*, which correspond to the cocycle*

$$b)\ \frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}.$$

2. *There exist two up to isomorphism $\mathfrak{s}$-even-homogeneous non-split supermanifolds with retract* $(-2,0,4)$. *The corresponding cocycles are*

$$a)\ z = \frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2}, \quad b)\ z = \frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x} - \frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1}.$$

*Proof.* Since $m=3$, the number of different pairs $i<j$ is less than or equal to 3. It follows from the Theorem 4 that $\dim H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{s}} \leq 3$. It is easy to see that $\dim H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{s}} = 3$ if and only if $k_1 = k_2 = k_3 = 2$. Let us take $A \in \operatorname{Aut}\mathbf{E} = \mathrm{GL}_3(\mathbb{C})$. Recall that $\operatorname{Int} A(z) = AzA^{-1}$. The direct calculation shows, see (12), that in the basis

$$v_1 = \tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x} + \tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1}, \quad v_2 = -\tfrac{1}{x}\xi_1\xi_3\tfrac{\partial}{\partial x} + \tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2}$$
$$v_3 = \tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x} + \tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3},$$

the automorphism $\operatorname{Int} A$ is given by

$$\operatorname{Int} A(v_i) = \det A \sum_j b_{ij} v_j. \tag{13}$$

Note that for any matrix $C \in \mathrm{GL}_3(\mathbb{C})$ there exists a matrix $B$ such that $C = \frac{1}{\det B}B$. Indeed, we can put $B = \frac{1}{\sqrt{\det C}}C$. Let us take a cocycle $z = \sum \alpha_i v_i \in H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{s}} \setminus \{0\}$. Obviously, it exists a matrix $D \in \mathrm{GL}_3(\mathbb{C})$ such that $D(z) = (0,0,1)$. Therefore, in the case $(2,2,2)$ there exists a unique up to isomorphism $\mathfrak{s}$-even-homogeneous non-split supermanifold given by the cocycle $\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}$.

Assume now that $\dim H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}=2$. Let us consider three cases.

**1.** Assume that $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$ is generated by two cocycles from the item 1 of Theorem 4. Obviously, we may consider only the case $k_1+k_2=4$, $k_1+k_3=4$. It follows that $k_2=k_3$. Denote $k_2:=k\neq 2$. Let us take $z\in H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}\setminus\{0\}$. Then $z=\frac{\alpha}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{k\alpha}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}+\frac{\beta}{x}\xi_1\xi_3\frac{\partial}{\partial x}+\frac{k\beta}{2x^2}\xi_1\xi_3\xi_2\frac{\partial}{\partial\xi_2}$. The group $\operatorname{Aut}\mathbf{E}$ contains in this case the subgroup $H$:

$$H:=\left\{\begin{pmatrix} a_{11} & 0 & 0\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{pmatrix}\right\}. \tag{14}$$

Let us take $A\in H$, denote $v_1:=-\frac{1}{x}\xi_1\xi_3\frac{\partial}{\partial x}+\frac{k}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$, $v_2:=\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{k}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}$. Using (12) or (13) we see that the operator $\operatorname{Int}A$ is given in the basis $v_1$, $v_2$ by:

$$\det A\begin{pmatrix} b_{22} & b_{23}\\ b_{32} & b_{33}\end{pmatrix}.$$

Obviously, for any cocycle $z=(-\beta,\alpha)\neq 0$ there exists a matrix $C\in\mathrm{GL}_3(\mathbb{C})$ such that $C(z)=(0,1)$. Therefore, in the case $(4-k,k,k)$ there exists a unique up to isomorphism $\mathfrak{s}$-even-homogeneous non-split supermanifold given by the cocycle $\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{k}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}$.

**2.** Assume that $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$ is generated by two cocycles from the item 2 of Theorem 4. We may consider only the case $k_1+k_2=2$, $k_1+k_3=2$, $k_2=k_3=0$. It follows that $(k_1,k_2,k_3)=(2,0,0)$. Let us take $z\in H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}\setminus\{0\}$. Then $z=\frac{\alpha}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}+\frac{\beta}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$, where $\alpha,\beta\in\mathbb{C}$. As above, the group $\operatorname{Aut}\mathbf{E}$ contains the subgroup $H$ given by (14). As above using the basis $v_1=\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$, $v_2=\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}$, we show that in the case $(2,0,0)$ there exists a unique up to isomorphism $\mathfrak{s}$-even-homogeneous non-split supermanifold given by the cocycle $\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$.

**3.** Assume that $H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}$ is generated by one cocycle from the item 1 and by one cocycle from the item 2 of Theorem 4. We may consider only the case $k_2+k_3=4$, $k_1+k_3=2$, $k_2=0$, i.e. $(k_1,k_2,k_3)=(-2,0,4)$. Let us take $z\in H^1(\mathbb{CP}^1,\mathcal{T}_2)^{\mathfrak{s}}\setminus\{0\}$. Then $z=\frac{\alpha}{x}\xi_2\xi_3\frac{\partial}{\partial x}-\frac{\alpha}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}+\frac{\beta}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$ for certain $\alpha,\beta\in\mathbb{C}$. Let us take $A\in\operatorname{Aut}\mathbf{E}$. Using Theorem 3 and 12, we get

$$\begin{aligned} A([\tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x}])A^{-1} &= [b_{11}\det A(\tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x}+(b_{12})'\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2})];\\ A([\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}])A^{-1} &= [b_{11}\det A\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}];\\ A([\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2}])A^{-1} &= [\det A(b_{22}\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2})],\end{aligned}$$

where $(b_{12})':=\frac{\partial}{\partial x}(b_{12})$. Consider the subgroup $H=\{\operatorname{diag}(a_{11},a_{22},a_{33})\}$ of $\operatorname{Aut}\mathbf{E}$. Let us choose the basis $v_1=\frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x}-\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}$, $v_2=\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}$

and take $A \in H$. Then the operator $\operatorname{Int} A$ is given by the matrix

$$(\det A) \operatorname{diag}(b_{11}, b_{22})$$

in the basis $v_1$, $v_2$. Obviously, for any cocycle $z = (\alpha, \beta) \neq 0$ there exists an operator $\operatorname{Int} A$ such that: $\operatorname{Int} A(z) = (1,1)$, if $\alpha \neq 0$, $\beta \neq 0$, $\operatorname{Int} A(z) = (0,1)$, if $\alpha = 0$, $\beta \neq 0$, $\operatorname{Int} A(z) = (1,0)$, if $\alpha \neq 0$, $\beta = 0$. Let us take

$$A = \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \operatorname{Aut} \mathbf{E}.$$

The direct calculation shows that $A(v_1)A^{-1} = v_1 + v_2$. In other words, $v_1$ and $v_1 + v_2$ corresponds to one orbit of the action Int. Since $b_{11} \neq 0$, we see that the cocycles $(0,1)$ and $(1,0)$ correspond to different orbits of the action Int.

In the case $\dim H^1(\mathbb{CP}^1, \mathcal{T}_2)^{\mathfrak{s}} = 1$ we may use the following proposition proven in [6].

**Proposition 1.** *If $\gamma \in H^1(\mathbb{CP}^1, \mathcal{T}_2)$ $c \in \mathbb{C} \setminus \{0\}$, then $\gamma$ and $c\gamma$ correspond to isomorphic supermanifolds.*

Theorem 7 follows.$\Box$

**Theorem 8.** [Classification of $\mathfrak{s}'$-even-homogeneous supermanifolds.] 1. *There exist two up to isomorphism $\mathfrak{s}'$-even-homogeneous non-split supermanifolds with retract*
*a)* $(2,2,1)$, *which corresponds to the cocycles*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{1}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}, \quad \frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2} - \frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1},$$

*b)* $(2,2,3)$, *which corresponds to the cocycles*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{3}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3},$$
$$\frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x} + \frac{1}{x^2}\xi_1\xi_3\frac{\partial}{\partial x} + \frac{2}{3x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1} - \frac{4}{3x^3}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2}.$$

2. *There exists a unique up to isomorphism $\mathfrak{s}'$-even-homogeneous non-split supermanifold with retract*
*a)* $(2,2,k)$, $k \neq 1,3$, *which corresponds to the cocycle*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x} + \frac{k}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3},$$

*b)* $(k,k,3-k)$, $k \neq 2$, *which corresponds to the cocycle*

$$\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2} - \frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1},$$

*c)* $(k,k,5-k)$, $k\neq 2$, *which corresponds to the cocycle*

$$\frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x}+\frac{1}{x^2}\xi_1\xi_3\frac{\partial}{\partial x}+\frac{k}{3x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}-\frac{2k}{3x^3}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}.$$

*d)* $(1,1,0)$, *which corresponds to the cocycle*

$$\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}.$$

*Proof.* By Theorem 5 and Proposition 1 we get 2.

Let us prove 1.$a$ Denote by $z$ a linear combination of cocycles (7). Let us take $A\in\operatorname{Aut}\mathbf{E}$. Using (12), we get:

$$A([\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x}])A^{-1}=[\det A(b_{33}\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x}+(b_{31})'\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}+(b_{32})'\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2})];$$

$$\begin{aligned}&A([\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_3}])A^{-1}=[\det A(b_{33}\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_3}+b_{32}\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2}+\\&+b_{31}\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1})];\\&A([\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2}])A^{-1}=[\det A(b_{21}\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}+b_{22}\tfrac{1}{x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2})],\\&A([\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}])A^{-1}=[\det A(b_{11}\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_1}+b_{12}\tfrac{1}{x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial\xi_2})].\end{aligned}$$

Consider the subgroup $H=\{\operatorname{diag}(a_{11},a_{11},a_{33},)\}$ of $\operatorname{Aut}\mathbf{E}$ and $A\in H$. Again a direct calculation shows that in the basis $v_1=\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{1}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}$, $v_2=\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}-\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}$ the automorphism $\operatorname{Int}A$ is given by $(\det A)\operatorname{diag}(b_{33},b_{11})$. Clearly, for $z=(\alpha,\beta)\neq 0$, there exist an operator $\operatorname{Int}A$ such that: $\operatorname{Int}A(z)=(1,1)$, if $\alpha\neq 0$, $\beta\neq 0$, $\operatorname{Int}A(z)=(0,1)$, if $\alpha=0$, $\beta\neq 0$, $\operatorname{Int}A(z)=(1,0)$, if $\alpha\neq 0$, $\beta=0$.

Let us take

$$A=\begin{pmatrix}1&0&0\\0&1&1\\-\frac{2}{3}x&2&1\end{pmatrix},$$

A direct calculation shows that $A(v_1+v_2)A^{-1}=v_1$. Since $b_{33}\neq 0$, we see that the cocycles $(0,1)$ and $(1,0)$ correspond to different orbits of the action Int. We have got 1$a$). The proof of 1$b$) is similar. The result follows.□

**Theorem 9.** [Classification of $\mathfrak{s}''$-even-homogeneous supermanifolds.] *There exists a unique up to isomorphism $\mathfrak{s}''$-even-homogeneous non-split supermanifold with retract* $(2,2,2)$, *which corresponds to the cocycle* (10); *and with retract* $(3,3,3)$, *which corresponds to the cocycle* (11).

*Proof.* It follows from Theorem 6 and Proposition 1.□

Comparing Theorems 7, 8 and 9, we get our main result:

**Theorem 10.** [Classification of even-homogeneous supermanifolds.]

1. *There exist two up to isomorphism even-homogeneous non-split supermanifolds with retract*

*a)* $(2,2,1)$, *which corresponds to the cocycles*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{1}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3},\quad \frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2}-\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1};$$

*b)* $(2,2,3)$, *which corresponds to the cocycles*

$$\begin{gathered}\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x}+\tfrac{3}{2x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3},\\ \tfrac{1}{x}\xi_2\xi_3\tfrac{\partial}{\partial x}+\tfrac{1}{x^2}\xi_1\xi_3\tfrac{\partial}{\partial x}+\tfrac{2}{3x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1}-\tfrac{4}{3x^3}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2};\end{gathered}$$

*c)* $(2,2,2)$, *which corresponds to the cocycles*

$$\begin{gathered}\tfrac{1}{x}\xi_1\xi_2\tfrac{\partial}{\partial x}+\tfrac{1}{2x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3},\\ \tfrac{1}{x^3}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_3}-\tfrac{1}{2x^2}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_2}+\tfrac{1}{2x}\xi_1\xi_2\xi_3\tfrac{\partial}{\partial \xi_1};\end{gathered}$$

*d)* $(-2,0,4)$, *which corresponds to the cocycles*

$$\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_2},\quad \frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x}-\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_1}.$$

2. *a) Assume that*

$$\{k,4-k,k_3\}\neq\{-2,0,4\},\ \{2,2,1\},\ \{2,2,3\},\ \{2,2,2\}.$$

*Then there exists a unique up to isomorphism even-homogeneous non-split supermanifold corresponding to the cocycle*

$$\frac{1}{x}\xi_1\xi_2\frac{\partial}{\partial x}+\frac{k_3}{2x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}.$$

*b) Assume that*

$$\{k,2-k,0\}\neq\{-2,0,4\}.$$

*Then there exists a unique up to isomorphism even-homogeneous non-split supermanifold corresponding to the cocycle*

$$\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial \xi_3}.$$

*There exists a unique up to isomorphism even-homogeneous non-split supermanifold with retract* *c)* $(k,k,3-k)$, $k\neq 2$, *which corresponds to the cocycle*

$$\frac{1}{x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}-\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1},$$

*d)* $(k,k,5-k)$, $k\neq 2$, *which corresponds to the cocycle*

$$\frac{1}{x}\xi_2\xi_3\frac{\partial}{\partial x}+\frac{1}{x^2}\xi_1\xi_3\frac{\partial}{\partial x}+\frac{k}{3x^2}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_1}-\frac{2k}{3x^3}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_2}.$$

*d)* $(3,3,3)$, *which corresponds to the cocycle (11).* $\Box$

By the similar argument as in [2], Corollary of Theorem 1, we get:
**Corollary.** Any non-split even-homogeneous supermanifold $\mathcal{M}$ of dimension $1|2$, where $\mathcal{M}_0=\mathbb{CP}^1$, is isomorphic to $\mathbb{Q}^{1|2}$.

Here $\mathbb{Q}^{1|2}$ is the (homogeneous) supermanifold corresponding to the cocycle $x^{-1}\xi_1\xi_2\frac{\partial}{\partial x}$ (see [2] for more details).

**Remark 1.** Theorem 10 gives rise to a description of even-homogeneous supermanifolds in the terms of local charts and coordinates. Indeed, let $\mathcal{M}$ be any supermanifold of dimension $1|m$, $m\leq 3$, with underlying space $\mathbb{CP}^1$, $v$ be the corresponding cocycle by Theorem 1 and $(U_0,\mathcal{O}_{\mathrm{gr}\,\mathcal{M}}|_{U_0})$, $(U_1,\mathcal{O}_{\mathrm{gr}\,\mathcal{M}}|_{U_1})$ be two standard charts of the retract $\mathrm{gr}\,\mathcal{M}$ with coordinates $(x,\xi_1,\xi_2,\xi_3)$ and $(y,\eta_1,\eta_2,\eta_3)$, respectively. In $U_0\cap U_1$ we have:

$$y=x^{-1},\quad \eta_i=x^{-k_i}\xi_i,\ i=1,2,3.$$

Consider an atlas on $\mathcal{M}$: $(U_0,\mathcal{O}_{\mathcal{M}}|_{U_0})$, $(U_1,\mathcal{O}_{\mathcal{M}}|_{U_1})$, with coordinates $(x',\xi'_1,\xi'_2,\xi'_3)$ and $(y',\eta'_1,\eta'_2,\eta'_3)$, respectively. Then the transition function of $\mathcal{M}$ in $U_0\cap U_1$ has the form

$$y'=(\mathrm{id}+v)(x'^{-1}),\quad \eta_i=(\mathrm{id}+v)((x')^{-k_i}\xi'_i),\ i=1,2,3.$$

**Remark 2.** The supermanifold $\mathcal{M}$ with the retract $(k,2-k,0)$, corresponding to the cocycle $\frac{1}{x}\xi_1\xi_2\xi_3\frac{\partial}{\partial\xi_3}$, which was lost in [2], is even-homogeneous but not homogeneous. Hence the main result in [2], Theorem 1, is correct.

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*Elizaveta Vishnyakova*
Mathematisches Forschungsinstitut Oberwolfach and
University of Luxembourg
*E-mail address:* `VishnyakovaE@googlemail.com`